

Unitary Howe dualities in fermionic and bosonic algebras and related Dirac operators

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Abstract

In this paper we use the canonical complex structure J on $\mathbb{R}^{2n}$ to introduce a twist of the symplectic Dirac operator. This can be interpreted as the bosonic analogue of the Dirac operators on a Hermitian manifold. Moreover, we prove that the algebra of these Dirac operators is isomorphic to the Lie algebra $\mathfrak{su}(1, 2)$ which leads to the Howe dual pair $(U(n), \mathfrak{su}(1, 2))$.



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1 Introduction

The CCR (canonical commuting relation) and CAR (canonical anticommuting relation) algebras are fundamental algebras in theoretical physics used for the study of bosons and fermions. From a mathematical viewpoint, these algebras are named the Weyl algebra (or symplectic Clifford algebra) and Clifford algebra. These algebras can be constructed in a very analogous way. The Clifford algebra is constructed on a vector space V equipped with a symmetric bilinear form B , whereas the Weyl algebra requires an even dimensional vector space equipped with a skew-symmetric bilinear form (or symplectic form) ω . In both cases, one then constructs the tensor algebra $T(V)$ where an ideal $I(V)$ is divided out. In the orthogonal setting, this is the ideal $I_B(V)$ with elements subject to the relation $\{u, v\} = 2B(u, v)$. In the symplectic setting, this is the ideal $I_\omega(V)$ generated by $[u, v] = -\omega(u, v)$.

There is, however, a fundamental difference: the Clifford algebra is finite-dimensional, whereas the Weyl algebra is infinite-dimensional. For the spinors (orthogonal versus symplectic) the same infinite-dimensional principle holds as for the Clifford algebras. As a matter of fact, the symplectic spinors are the smooth vectors in the metaplectic representation [2]. Using the generators of the Clifford (resp. Weyl) algebra, one can associate a natural first order spin (resp. metaplectic) invariant differential operator by contracting the Clifford algebra elements using the bilinear form B (resp. the symplectic form ω) with derivatives. This gives rise to

the Dirac operator $\partial_x = \sum_{k=1}^n e_k \partial_{x_k}$ where $\{e_j, e_k\} = -2\delta_{ij}$ and the symplectic Dirac operator $\sum_{k=1}^n (iq_k \partial_{y_k} - \partial_{q_k} \partial_{x_k})$ where $[\partial_{q_j}, iq_k] = i\delta_{jk}$ are the Heisenberg relations (see [1, 2, 4]).

The theory which studies the solutions of the Dirac operator is known as Clifford analysis and can be seen as a hypercomplex function theory. Moreover, quite some generalisations have occurred in the last two decades. This involves e.g. Clifford analysis on superspace and Clifford analysis on (hyper)Kähler spaces. It is in the latter framework in which this paper is situated, but then from a symplectic point of view. More precisely, we provide the foundations of what we will call a *Hermitian variant of symplectic Clifford analysis*, where we incorporate the additional datum of a compatible complex structure $\mathbb{J}$ on the flat symplectic space $\mathbb{R}^{2n}$. This leads to the study of the symplectic Dirac operator on a Kähler manifold as was already initiated in [2, 3]. However, the underlying invariance symmetry and the algebra generated by these type of operators (and their duals) was never investigated.

2 Rudiments of symplectic Clifford analysis

Let us consider the canonical symplectic space $\mathbb{R}^{2n}$ with coordinates (x, y) and the usual symplectic form $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$ which has the matrix representation $\Omega_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Recall that the symplectic group $\text{Sp}(2n, \mathbb{R})$ is the group given by invertible linear transformations preserving the non-degenerate skew-symmetric bilinear form from above and is given (in terms of matrices) by

$$\text{Sp}(2n, \mathbb{R}) = \{M \in \text{GL}(2n, \mathbb{R}) \mid M^T \Omega_0 M = \Omega_0\}.$$

The group is non-compact and has dimension $2n^2 + n$. Moreover, the corresponding Lie algebra is denoted by $\mathfrak{sp}(2n, \mathbb{R})$. The main difference with the orthogonal case, lies in the fact that the metaplectic group (the double cover of the symplectic group) does *not* admit a finite-dimensional representation (it is not a matrix group). This is a strong contrast with the spin representation in the orthogonal case. Moreover, the orthogonal spinors $\mathbb{S}$ are realised as a idempotent left ideal in the Clifford algebra, which is not the case for the symplectic spinors. As mentioned, the symplectic equivalent of the spin representations are infinite dimensional, which means that one needs to work with the theory of unitary representations.

2.1 The Schwartz space and metaplectic representation

For further convenience, we fix notation and define the Schwartz space, which plays a crucial role in the construction of the metaplectic representation. On the space $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ we define (using the multi-index notation) the norm $\|f\|_{\alpha, \beta} := \sup_{q \in \mathbb{R}^n} |q^\alpha (D^\beta f)(q)|$ for all $\alpha, \beta \in \mathbb{N}^n$. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the subspace of $L^p(\mathbb{R}^n)$ (for $1 \leq p \leq \infty$) consisting of rapidly decreasing functions and is given by

$$\mathcal{S}(\mathbb{R}^n, \mathbb{C}) := \{f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}) : \|f\|_{\alpha, \beta} < \infty, \text{ for all } \alpha, \beta \in \mathbb{N}^n\}.$$

We now describe (following [2]) the infinite-dimensional Segal-Shale-Weil representation (also oscillator or metaplectic representation) of the metaplectic group. The smooth vectors of the unitary representation $m : \text{Mp}(2n) \rightarrow \text{U}(L^2(\mathbb{R}^n))$ coincide with the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and are a model for the symplectic spinors $\mathbb{S}^\infty$. Due to Stone-Von Neumann theorem the representation is unique (up to unitary equivalence).

2.2 The symplectic Clifford algebra and the related Dirac operator

Let (V, ω) be a symplectic vector space. The *symplectic Clifford algebra* $\text{Cl}_s(V, \omega)$ is defined as the quotient algebra of the tensor algebra $T(V)$ of V , by the two-sided ideal

$$\mathcal{I}_\omega := \{v \otimes u - u \otimes v + \omega(v, u) : u, v \in V\}.$$

In other words $\text{Cl}_s(V, \omega) := T(V)/\mathcal{I}_\omega$ is the algebra generated by V in terms of the relation $[v, u] = -\omega(v, u)$, where we have omitted the tensor product symbols. We refer to the symplectic Clifford algebra on $\mathbb{R}^{2n}$ as the n th Weyl algebra $\mathcal{W}_n$ with generators $iq_1, \dots, iq_n, \partial_{q_1}, \dots, \partial_{q_n}$ satisfying the commutation relations $[q_j, q_k] = 0$ and $[\partial_{q_j}, q_k] = \delta_{jk}$.

Denote by $\mathcal{F}$ a suitable function space (e.g. the space of polynomials, or smooth functions). The *symplectic Dirac operator* on $(\mathbb{R}^{2n}, \omega_0)$ is the first-order (in the base variables x and y) differential operator acting on a symplectic spinor-valued functions space $\mathcal{F} \otimes \mathbb{S}^\infty$ given by $D_s = \sum_{j=1}^n (iq_j \partial_{y_j} - \partial_{q_j} \partial_{x_j})$. With respect to the symplectic Fischer inner product (see [4]), we obtain the dual operator $X_s = \sum_{j=1}^n (iq_j x_j + \partial_{q_j} y_j)$. These operators satisfy the relations:

$$\begin{aligned} [\mathbb{E} + n, X_s] &= X_s, \\ [\mathbb{E} + n, D_s] &= -D_s, \\ [D_s, X_s] &= -i(\mathbb{E} + n), \end{aligned}$$

where $\mathbb{E} = \sum_{j=1}^n (x_j \partial_{x_j} + y_j \partial_{y_j})$ is the *Euler operator*. In other words, the three operators give rise to a copy of the Lie algebra $\mathfrak{sl}(2)$.

3 The interaction with a complex structure

3.1 Definition of the twisted symplectic Dirac operators

We will now introduce a complex structure $\mathbb{J}$ on the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$ which is compatible with the symplectic form ω_0 . This means that $\omega_0(x, \mathbb{J}y)$ defines a Riemannian metric g . Otherwise said, we will be working with the canonical Kähler manifold $(\mathbb{R}^{2n}, \omega_0, g, \mathbb{J})$. By Darboux's theorem, we obtain, with respect to the canonical symplectic basis $\{e_j\}_{j=1}^{2n}$ the following complex structure $\mathbb{J} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. This means that the action of the complex structure $\mathbb{J}$ on $\mathbb{R}^{2n}$ is given by

$$(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto (y_1, \dots, y_n, -x_1, \dots, -x_n).$$

The new differential operators acting on symplectic spinor-valued functions

$$\tilde{D}_s = \sum_{j=1}^n iq_j \partial_{x_j} + \partial_{y_j} \partial_{q_j}, \quad \tilde{X}_s = \sum_{j=1}^n x_j \partial_{q_j} - iy_j \partial_{x_j}, \quad \mathbb{E} = \sum_{j=1}^n x_j \partial_{x_j} + y_j \partial_{y_j},$$

also give rise to a copy of the Lie algebra $\mathfrak{sl}(2)$. We call these first two operators the *twists* of D_s and X_s . Both sets of operators, i.e. (D_s, X_s) and $(\tilde{D}_s, \tilde{X}_s)$, are symplectic invariant, albeit under the following two *different* realisations of the symplectic Lie algebra given by:

$$\left\{ \begin{aligned} X_{jk} &= x_j \partial_{x_k} - y_k \partial_{y_j} - (q_k \partial_{q_j} + \frac{1}{2} \delta_{jk}), \\ Y_{jk} &= x_j \partial_{y_k} + x_k \partial_{y_j} + i \partial_{q_j} \partial_{q_k}, \\ Z_{jk} &= y_j \partial_{x_k} + y_k \partial_{x_j} + iq_j q_k, \\ Y_{jj} &= x_j \partial_{y_j} + \frac{i}{2} \partial_{q_j}^2, \\ Z_{jj} &= y_j \partial_{x_j} + \frac{i}{2} \partial_{q_j}^2, \end{aligned} \right. \quad \text{and} \quad \left\{ \begin{aligned} \tilde{X}_{jk} &= x_j \partial_{x_k} - y_k \partial_{y_j} + q_k \partial_{q_j} + \frac{1}{2} \delta_{jk}, & 1 \leq j \leq k \leq n, \\ \tilde{Y}_{jk} &= x_j \partial_{y_k} + x_k \partial_{y_j} - iq_j q_k, & j < k = 1, \dots, n, \\ \tilde{Z}_{jk} &= y_j \partial_{x_k} + y_k \partial_{x_j} - i \partial_{q_j} \partial_{q_k}, & j < k = 1, \dots, n, \\ \tilde{Y}_{jj} &= x_j \partial_{y_j} - \frac{i}{2} \partial_{q_j}^2, & j = 1, \dots, n, \\ \tilde{Z}_{jj} &= y_j \partial_{x_j} - \frac{i}{2} \partial_{q_j}^2, & j = 1, \dots, n. \end{aligned} \right.$$

Of course, it is not very useful that D_s and $\tilde{D}_s$ are invariant under different (yet isomorphic) $\mathfrak{sp}(2n, \mathbb{R})$ -realisations. Therefore, we will perform a symmetry reduction so that both operators become invariant under *one and the same* Lie algebra. To that end, we need to find the symplectic matrices which commute with the complex structure. We claim that

$$\mathrm{Sp}_{\mathbb{J}}(2n, \mathbb{R}) := \{M \in \mathrm{Sp}(2n, \mathbb{R}) \mid M\mathbb{J} = \mathbb{J}M\},$$

defines a realisation for the unitary Lie group. In order to see this, assume that M is of the block-form: $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C and D are $(n \times n)$ -matrices. The condition that M is symplectic is equivalent to the one of the following conditions: the matrices $A^T C$ and $B^T D$ are symmetric and $A^T D - C^T B = I$. So, in order to determine $\mathrm{Sp}_{\mathbb{J}}(2n, \mathbb{R})$ we need to determine the symplectic matrices M which commute with the complex structure $\mathbb{J}$. The latter conditions means that

$$\begin{aligned} M\mathbb{J} = \mathbb{J}M &\iff \mathbb{J}^{-1}M\mathbb{J} = M \\ &\iff \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &\iff \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \end{aligned}$$

This implies $A = D$ and $B = -C$. In other words the matrix M is of the form: $M = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. Next, we still have the condition that M is symplectic, i.e.

$$\begin{aligned} M^T \Omega M = \Omega &\iff \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \\ &\iff \begin{pmatrix} -C^T & A^T \\ -D^T & B^T \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -C^T A + A^T C & -C^T B + A^T D \\ -D^T A + B^T C & -D^T B + B^T D \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \\ &\iff \begin{cases} A^T C = C^T A, \\ A^T D - C^T B = I, \\ B^T C - D^T A = -I, \\ B^T D = D^T B. \end{cases} \end{aligned}$$

This means that $A^T C$ and $B^T D$ should be symmetric matrices and $A^T D - C^T B = I$. But now, due to the first condition this reduces to $B^T A = A^T B$ and $A^T A + B^T B = I$. In other words, the matrices we are looking for must be of the form $M = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ with $B^T A = A^T B$ and $A^T A + B^T B = I$, i.e.

$$\begin{pmatrix} A^T & -B^T \\ B^T & A^T \end{pmatrix} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} A^T A + B^T B & A^T B - B^T A \\ B^T A - A^T B & B^T B + A^T A \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Which is exactly the condition for a unitary matrix. The map

$$\Phi : \mathrm{Sp}_{\mathbb{J}}(2n, \mathbb{R}) \rightarrow \mathrm{U}(n) : M \mapsto A + iB,$$

gives the wanted isomorphism.

3.2 Unitary invariant symplectic Dirac operators

One can now check that the symplectic Dirac operator and its twist, are unitary invariant differential operators. This can be done by verifying that the operators commute with the

following realisation of unitary Lie algebra $u(n)$:

$$\begin{cases} A_{jk} = y_j \partial_{x_k} + y_k \partial_{x_j} - x_j \partial_{y_k} - x_k \partial_{y_j} + i(q_j q_k - \partial_{q_j} \partial_{q_k}), & 1 \leq j < k \leq n, \\ B_{jj} = y_j \partial_{x_j} - x_j \partial_{y_j} + \frac{i}{2} (q_j^2 - \partial_{q_j}^2), & 1 \leq j \leq n, \\ C_{jk} = x_j \partial_{x_k} - x_k \partial_{x_j} + y_j \partial_{y_k} - y_k \partial_{y_j} + q_j \partial_{q_k} - q_k \partial_{q_j}, & 1 \leq j < k \leq n. \end{cases}$$

This means that we can refine the $\mathfrak{sp}(2n)$ -invariant PDE $D_s f = 0$ into two $u(n)$ -invariant PDEs given by $D_s f = 0$ and $\tilde{D}_s f = 0$, for a symplectic spinor valued polynomial $f \in \mathcal{P}(\mathbb{R}^{2n}, \mathbb{C}) \otimes S(\mathbb{R}^n)$. In analogy with the orthogonal case, we call the solutions *hermitian symplectic monogenics* (or *h-symplectic monogenics* in short).

3.3 Symplectic Dolbeault operators

Moreover, there is a second way of introducing the twist of the symplectic Dirac operator. Let us define the following operators which are known in the literature as the *symplectic Dolbeault operators* [3] defined by means of $D_z = \frac{D_s + i\tilde{D}_s}{2}$ and $D_z^\dagger := \frac{D_s - i\tilde{D}_s}{2}$. One easily verifies that

$$\frac{1}{2}(D_s + i\tilde{D}_s) = -\sum_{j=1}^n \mathfrak{F}_j \partial_{z_j}, \quad \text{and} \quad \frac{1}{2}(D_s - i\tilde{D}_s) = \sum_{j=1}^n \mathfrak{F}_j^\dagger \partial_{\bar{z}_j},$$

where we have introduced the symbols $\mathfrak{F}_j = (q_j + \partial_{q_j})$, $\mathfrak{F}_j^\dagger = (q_j - \partial_{q_j})$ and $\partial_{z_j} := \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ is the Cauchy-Riemann operator in the relevant variable with conjugate $\partial_{\bar{z}_j} := \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$. The structure of the operators D_z and $D_z^\dagger$ is similar to the orthogonal case. However, the raising/lowering operators $\mathfrak{F}_j$ and $\mathfrak{F}_j^\dagger$ are used instead of isotropic Witt vectors $\mathfrak{f}_j$ and $\mathfrak{f}_j^\dagger$ (see for instance [5] and the references therein).

3.4 Class of simultaneous solutions of D_s and $\tilde{D}_s$

We will now describe a wide class of examples of *h-symplectic monogenics*, by making the link with holomorphic functions in several variables. Let $f : \Omega \subset \mathbb{C}^n \rightarrow \mathbb{C}$ be a complex-valued function in several complex variables which is of the class $\mathcal{C}^1(\Omega)$ (i.e. continuously differentiable). We say that f is holomorphic (in several variables) if $\partial_{\bar{z}_j} f(z_1, \dots, z_n) = 0$ for all $1 \leq j \leq n$. Moreover, we denote the set of holomorphic functions in Ω by $\text{Hol}(\Omega)$.

In order not to overload notations, we use the summation convention. Suppose that we have a function of the form $F(x, y, q) = e^{-\frac{1}{2}|q|^2} H(x, y)$. Letting the symplectic Dirac operator act on F gives:

$$\begin{aligned} D_s \left(e^{-\frac{1}{2}|q|^2} H(x, y) \right) &= (iq_k \partial_{y_k} - \partial_{x_k} \partial_{q_k}) \left(e^{-\frac{1}{2}|q|^2} H(x, y) \right) \\ &= iq_k e^{-\frac{1}{2}|q|^2} \partial_{y_k} H(x, y) + e^{-\frac{1}{2}|q|^2} q_k \partial_{x_k} H(x, y) \\ &= e^{-\frac{1}{2}|q|^2} q_k (\partial_{x_k} + i\partial_{y_k}) H(x, y). \end{aligned}$$

We note that this equals zero if $(\partial_{x_k} + i\partial_{y_k}) H(x, y) = 0$ for all $k = 1, \dots, n$, i.e. if $H(x, y)$ is a holomorphic function in several variables. Completely similar,

$$\begin{aligned} \tilde{D}_s \left(e^{-\frac{1}{2}|q|^2} H(x, y) \right) &= (iq_k \partial_{x_k} + \partial_{y_k} \partial_{q_k}) \left(e^{-\frac{1}{2}|q|^2} H(x, y) \right) \\ &= e^{-\frac{1}{2}|q|^2} q_k (i\partial_{x_k} - \partial_{y_k}) H(x, y) \\ &= ie^{-\frac{1}{2}|q|^2} q_k (\partial_{x_k} + i\partial_{y_k}) H(x, y), \end{aligned}$$

which is zero for holomorphic $H(x, y)$.

This means that every function of the form $e^{-\frac{1}{2}(q_1^2+\dots+q_n^2)}H(x,y)$ with H an holomorphic function in several variables is a solution of both D_s and $\tilde{D}_s$. This observation generalises the class of solutions obtained by Habermann in [2] for $n = 1$. It turned out that there are much *more solutions* than the ones of this form. In order to describe these systematically, we will need the notion of Howe dualities and corresponding Fischer decompositions. This will be done in full detail in our upcoming paper [6]. In the following section we will reveal the algebraic structures required for this approach.

4 A unitary Howe duality associated with D_s and $\tilde{D}_s$

Recall that the Lie algebra $\mathfrak{su}(1,2)$ is a quasi-split real form of the complex Lie algebra $\mathfrak{sl}(3)$ and is defined in terms of matrices as

$$\mathfrak{su}(1,2) = \left\{ \begin{pmatrix} \alpha & \beta & ic \\ \gamma & \bar{\alpha}-\alpha & -\bar{\beta} \\ id & -\bar{\gamma} & -\bar{\alpha} \end{pmatrix} \mid c,d \in \mathbb{R} \text{ \& } \alpha,\beta,\gamma \in \mathbb{C} \right\}.$$

A basis of the Lie algebra $\mathfrak{su}(1,2)$ is given by the the following eight matrices:

$$\begin{aligned} H_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & i \end{pmatrix}, \\ X_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ Y_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}. \end{aligned}$$

The commutation relations of these matrices give rise to the following table:

$[\cdot, \cdot]$	H_1	H_2	X_1	X_2	X_3	Y_1	Y_2	Y_3
H_1	0	0	X_1	X_2	$2X_3$	$-Y_1$	$-Y_2$	$-2Y_3$
H_2		0	$3X_2$	$-3X_1$	0	$-3Y_2$	$3Y_1$	0
X_1			0	$2X_3$	0	H_1	H_2	$-Y_2$
X_2				0	0	H_2	$-H_1$	$-Y_1$
X_3					0	$-X_2$	$-X_1$	$-H_1$
Y_1						0	$-2Y_3$	0
Y_2							0	0
Y_3								0

The Lie algebra generated by the symplectic Dirac operators $D_s, \tilde{D}_s$ and their duals $X_s, \tilde{X}_s$ gives rise to a copy of the Lie algebra $\mathfrak{su}(1,2)$. In order to close the algebra, we introduce the following differential operators:

$$\mathcal{O} := \sum_{j=1}^n i(x_j \partial_{y_j} - y_j \partial_{x_j}) + \partial_{q_j}^2 - q_j^2, \quad \Delta := \sum_{j=1}^n \partial_{x_j}^2 + \partial_{y_j}^2, \quad \text{and} \quad r^2 := \sum_{j=1}^n x_j^2 + y_j^2.$$

It follows from straightforward calculations that the eight operators can be identified with the matrices from above: $H_1 \leftrightarrow \mathbb{E}, H_2 \leftrightarrow \mathcal{O}, X_1 \leftrightarrow X_t, X_2 \leftrightarrow X_s, X_3 \leftrightarrow r^2, Y_1 \leftrightarrow \tilde{D}_s, Y_2 \leftrightarrow D_s$ and $Y_3 \leftrightarrow \Delta$.

1. We have two copies of the Heisenberg algebra: $\text{Alg}\{D_s, \tilde{D}_s, \Delta\} \cong \text{Alg}\{X_s, \tilde{X}_s, r^2\} \cong \mathfrak{h}_3$.
2. We have three copies of $\mathfrak{sl}(2)$: $\text{Alg}\{D, D^\dagger, \mathbb{E}\} \cong \text{Alg}\{\tilde{D}, \tilde{D}^\dagger, \mathbb{E}\} \cong \text{Alg}\{\Delta, r^2, \mathbb{E}\} \cong \mathfrak{sl}(2)$.

This means that there is a canonical $\mathfrak{su}(1, 2)$ -action on the space of spinor valued polynomials $\mathcal{P}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$ where restricting to the subalgebra $\text{Alg}(D_s, X_s)$ corresponds to $\mathfrak{sl}(2)$ -copy obtained in the Howe duality for symplectic Clifford analysis (see [4] for more details). Now, taking into account the symplectic Dirac operators and its twists, we obtain the dual pair $U(n) \times \mathfrak{su}(1, 2)$ (i.e. the underlying group of invariance, together with the algebra generated by the operators and their duals).

We now focus on the reduction of the symplectic spinor space. In the orthogonal case, the spinor space $\mathbb{S}$ decomposes under the action of the unitary group $U(n)$ as $\mathbb{S} = \bigoplus_r \mathbb{S}_{(r)}$, with $\mathbb{S}_{(r)}$ inequivalent irreducible pieces, which are eigenspaces of the *fermionic* quantum oscillator (also called spin-Euler operator, see for instance [5]). In the symplectic case, the relevant operator for decomposing the infinite dimensional spinor space is the *bosonic* quantum oscillator. The Hamiltonian of the quantum oscillator, the so-called Hermite operator, is given by

$$\mathcal{H} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n), \quad f(q) \mapsto \frac{1}{2} \sum_{j=1}^n (\partial_{q_j}^2 - q_j^2) f(q).$$

Note that we can write $\mathcal{O} = \sum_{j=1}^n i(x_j \partial_{y_j} - y_j \partial_{x_j}) + 2\mathcal{H}$, so that the Hermite operator is in fact the spinor-valued part of the operator $\mathcal{O}$, i.e. the differential operator in ∂_{q_j} and the variables q_j . Moreover, the eigenspaces can be identified with the irreducible decomposition of $\mathbb{S}^\infty$ into $u(n)$ -irreducible representations. This means that the symplectic spinor space $\mathbb{S}^\infty$ decomposes into $u(n)$ -irreducible representations $\tilde{\mathbb{S}}_{(k)}^\infty$ of dimension $\binom{n+k-1}{k}$ which can be thought of as k -homogeneous polynomials or the eigenspaces of the Hermite operator $\mathcal{H}$.

Moreover, the solutions of the corresponding Dirac operators, called monogenics, can be introduced from a purely representation theoretical viewpoint. In general, this boils down to determining the decomposition (this is called a *Fischer decomposition*) $\mathcal{P}_k(\mathbb{R}^m, \mathbb{C}) \otimes \mathbb{S}$ where $\mathbb{S}$ is the spinor space, which is $\mathbb{S}$ in the orthogonal case and $\mathbb{S}^\infty$ in the symplectic case, where we take $m = 2n$ in particular. We denote by $\mathcal{M}_k$ the k -homogeneous solutions of the Dirac operator $\partial_x = \sum_{j=1}^m e_j \partial_{x_j}$, these are called monogenics. They can be defined as follows:

$$\mathcal{M}_k \leftrightarrow (k, 0, \dots, 0) \boxtimes \mathbb{S} = (k) \boxtimes \left(\frac{1}{2}, \dots, \pm \frac{1}{2} \right) \cong \left(k + \frac{1}{2}, \dots, \pm \frac{1}{2} \right),$$

where $\boxtimes$ denotes the Cartan product of the $\mathfrak{so}(m)$ -representations and $(k, 0, \dots, 0)$ is the space of k -homogeneous harmonics (solutions of the Laplacian). In the symplectic case, we have:

$$\begin{aligned} \mathcal{M}_k^s \leftrightarrow (k, 0, \dots, 0)_s \boxtimes \mathbb{S}^\infty &= (k)_s \boxtimes \left(\left(-\frac{1}{2}, \dots, -\frac{1}{2} \right) \oplus \left(-\frac{1}{2}, \dots, -\frac{3}{2} \right) \right) \\ &\cong \left(k - \frac{1}{2}, \dots, -\frac{1}{2} \right) \oplus \left(k - \frac{1}{2}, \dots, -\frac{3}{2} \right), \end{aligned}$$

where $(k, 0, \dots, 0)_s$ is the space of k -homogeneous polynomials. In order to obtain an algebraic characterisation of the space of h -symplectic monogenics, one proceeds as follows. First of all, we note that we need to consider the symplectic spinors $\mathbb{S}^\infty$ from an unitary viewpoint. We saw that $\mathbb{S}^\infty$ decomposes as an infinite direct sum of finite dimensional $u(n)$ -modules $\tilde{\mathbb{S}}_{(k)}^\infty$ which are in fact eigenspaces of the Hermite operator. We denote the branched spinor space (which is in fact a direct sum of $u(n)$ -irreps) by $\widetilde{\mathbb{S}^\infty}$. However, the space of k -homogeneous polynomials is *not* irreducible as a $u(n)$ -module and we denote the branched module by $\widetilde{(k)}$. This means that the Cartan product $\mathcal{M}_k^{hs} \leftrightarrow \widetilde{(k)} \boxtimes \widetilde{\mathbb{S}^\infty}$ would be a well-educated guess as a representation

theoretical definition of the h -symplectic monogenics. Recall that these are the symplectic spinor-valued polynomial functions $f \in \mathcal{P}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$ such that $D_s f = \tilde{D}_s f = 0$. The explicit calculation of the Cartan product (and more generally the tensor product) will be done in [6]. Moreover, as an application we will prove a Fischer decomposition for the Howe dual pair we obtained in this paper.

5 Conclusion

In this paper we investigated a Howe dual pair occurring in symplectic Clifford analysis by allowing a compatible complex structure. This Howe duality is of the form $(G, \mathfrak{g}')$ where G is the underlying invariance group for which the relevant Dirac operators are invariant and $\mathfrak{g}'$ is the algebra generated by the Dirac operators and their duals. Depending on the orthogonal or symplectic framework, we have the following ‘types’ of Clifford analysis and *refinements*:

1. Orthogonal geometry (giving rise to a Clifford algebra)
 - (a) Clifford analysis: $SO(n) \times \mathfrak{osp}(1|2)$
 - (b) Hermitian Clifford analysis $U(n) \times \mathfrak{osp}(2|2)$
 - (c) Quaternionic Clifford analysis $USp(n) \times \mathfrak{osp}(4|2)$
2. Symplectic geometry (giving rise to a Weyl algebra)
 - (a) Symplectic Clifford analysis: $Sp(2n) \times \mathfrak{sl}(2)$
 - (b) Hermitian symplectic Clifford analysis: $U(n) \times \mathfrak{su}(1, 2)$
 - (c) Quaternionic symplectic Clifford analysis: $USp(n) \times ?$

Thus far, we extended the framework of hermitian Clifford analysis in the presence of a symplectic structure in the case of the (flat) Kähler manifold $\mathbb{R}^{2n}$. It is an interesting question to further reduce the symmetry to the compact symplectic group $USp(n)$ so that we have the chain $Sp(2n) \supset U(n) \supset USp(n)$. In our future work [6], we will describe the Fischer decomposition accompanying this new Howe dual pair.

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